

On anomalous distributions in intra-day financial time series and Non-extensive Statistical Mechanics

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Abstract

In this paper one studies the distribution of log-returns (tick-by-tick) in the Lisbon stock market and shows that it is well adjusted by the solution of the equation, $\frac{dp_x}{d|x|} = -\beta_{q'} p_x^{q'} - (\beta_q - \beta_{q'}) p_x^q$, which corresponds to a generalization of the differential equation which has as solution the power-laws that optimise the entropic form $S_q = -k \frac{1 - \int p_x^q dx}{1-q}$, base of present non-extensive statistical mechanics.

Key words: Econophysics, Nonextensive Statistical Mechanics, Complex systems
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1 Introduction

Since Mandelbrot's work [1] on analysing cotton prices that Lévy distribution has been widely used, but, frequently with Lévy scaling coefficients greater than 2, and so, out of Lévy regime[2,3]. An alternative way of describing these tailed distributions can be power laws (usually called *q-Gaussians* or *Tsallis distributions*) obtained by optimising the entropic form proposed by Constantino Tsallis[4,5,6]. Those distributions, beyond presenting the advantage of finite second (and higher) order momenta (for a certain range of values of its characterizing index q) are also the solution of various kinds of non-linear Fokker-Planck equations[7,8,9] often used as continuous time approaches to financial systems. In this manuscript one shows that a generalisation of non-extensive formalism permits a good adjustment to the tick-by-tick log-return distributions for Lisbon Stock Market.

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2 From Boltzmann-Gibbs-Shannon to anomalous distribution

It is known that the consecrated Boltzmann-Gibbs-Shannon (BGS) entropic form, $S = -\int p(x) \log p(x) dx$, is able to provide a perfect description of an all range of phenomena with microscopic spatial/temporal short-range interactions, but is not capable to give proper answers when systems present microscopic spatial/temporal long-range interactions [11,10] or multifractal structure [12]. In order to deal with this glitch, Tsallis presented the entropic form[13],

$$S_q = -k \frac{1 - \int [p(x)]^q dx}{1 - q} \quad (1)$$

($q \in \mathbb{R}$), which contains BGS as a particular case when $q = 1$. For this entropic form, the optimisation, under the appropriate constraints $\int p(x) dx = 1$ (unitary norm) and $\int x^2 \frac{[p(x)]^q}{\int [p(x)]^q dx} dx = \sigma_q^2$ (finite generalised variance), yields the following power-law (q -Gaussian),

$$p(x) \sim \left[1 - \beta_q (1 - q) x^2\right]^{\frac{1}{1-q}} \equiv e_q^{-\beta_q x^2}, \quad (2)$$

which in the limit $q \rightarrow 1$ recovers the Gaussian distribution. The q -Gaussian, $e_q^{-\beta_q x^2}$, can be rewritten, in a quite interesting way, as the solution for the non-linear differential equation,

$$\frac{dp(x)}{d(x^2)} = -\beta_q [p(x)]^q. \quad (3)$$

From this equation one can obtain the (simmetrical) q -exponential distribution changing x^2 by $|x|$ and the exponential distribution considering also $q \rightarrow 1$. For reasons that will be presented in the next section let one extends Eq. (3), for the q -exponential case, in the same way that was done in previous works on protein re-association [15] and flux of cosmic rays, by introducing another index q' ,

$$\frac{dp(x)}{d|x|} = -\beta_{q'} [p(x)]^{q'} - (\beta_q - \beta_{q'}) [p(x)]^q. \quad (4)$$

(It is interesting to notice that if we consider $q' = 1$ and $q = 2$ one obtains the expression used by Max Planck in 1900 that lead him to the black-body radiation law [17]). The solution of Eq. (4) yields,

$$|x| = \frac{1}{\beta_{q'}} \left\{ \frac{[p(x)]^{-(q'-1)}}{q' - 1} - \frac{(\beta_q/\beta_{q'}) - 1}{1 + q - 2q'} \times \right. \quad (5)$$

$$\left. \times [J(1; q - 2q', q - q', (\beta_q/\beta_{q'}) - 1) - J(p(x); q - 2q', q - q', (\beta_q/\beta_{q'}) - 1)] \right\}$$

where $J(p(x); a, b, c) = [p(x)]^{1+a} F\left(\frac{1+a}{b}; \frac{1+a+b}{c}; [p(x)]^b c\right)$ and F the hypergeometric function. For $1 < q' < q$ and $\beta_{q'} \ll \beta_q$ three regions can be observed. The first one (where q governs) for $0 \leq |x| \ll |x|^*$, where $|x|^* = \frac{1}{(q-1)\beta_q}$, a second one (where both q and q' govern) for $|x|^* \ll |x| \ll |x|^{**} \equiv \frac{[(q-1)\beta_q]^{\frac{q'-1}{q-q'}}}{[(q'-1)\beta_{q'}]^{\frac{q-1}{q-q'}}}$ and the third one (governed by q') for $|x| \gg |x|^{**}$.

3 Application to Lisbon Stock Market

Let one now considers the variable x_i as the tick-by-tick log-return obtained by, $x_i = \log \frac{\Pi_i}{\Pi_{i-1}}$, with Π_i representing the value of stock market index at tick i . Here one neglects the time spacing between transactions which are, in mean, of order of 10 seconds and its possible influence in the magnitude of variations. The description of log-return's distribution has been successfully done with q -Gaussians when the lag between elements at least of the order of minutes (or greater), but fails when one deals with time series where lag is smaller for which data seems to be better approached by exponential functions [14]. However, as can be seen in Fig. 1, for the case of the tick-by-tick return in the Lisbon stock market the exponential function it is only able to fit the central part of the distribution. By solving numerically the Eq.(4) with $q' = 1.076$, $q = 1.534$, $\beta_{q'} = 6.59 \times 10^3$ and $\beta_q = 7.47 \times 10^4$ one obtained the full line which presents a very good agreement, and clearly the best compared to other distributions, for data from Lisbon stock market for six decades in ordinate and two and a half decades in abscissa. For the values q, q', β_q and $\beta_{q'}$ used, $|x|^* \simeq 0.00003$ and $|x|^{**} \simeq 0.004$. Unfortunately, the finiteness of data does not allow the observation of the second crossover which occurs at a value of x where it is not possible to have a good statistics.

4 Concluding remarks

In conclusion one can state that like in biological and cosmological phenomena, the superposition of power-laws which maximise Tsallis' entropy can be an excellent approach in order to describe anomalous distributions in finance, fact that could be a sign of a peculiar dynamics (probably due to market characteristics like volume, number of traders) different than the ones presented

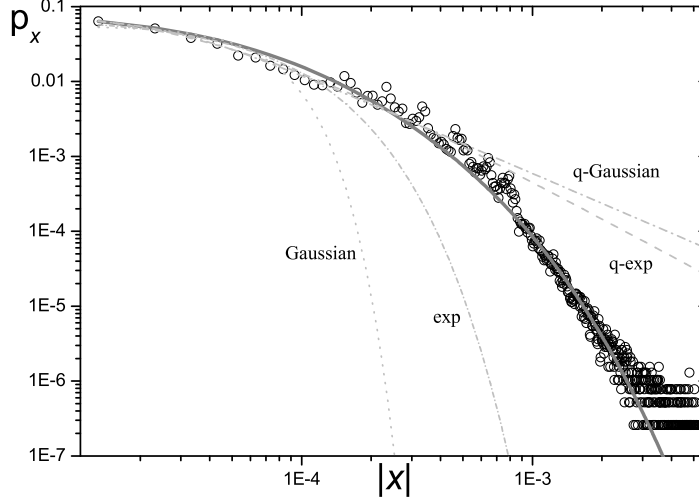


Fig. 1. Probability density function (PDF) for the tick by tick log-return in the Lisbon Stock Market between 1st Feb. 1996 to 28th Jun. 2002 (4 million ticks approx.). The full line represents the solution of Eq. (4) and it is clearly best approach to PDF from data (circles) when compared with the best Gaussian, q -Gaussian ($q = 2.51$), exponential and q -exponential ($q = 1.59$) fits that are also plotted.

until now. Despite the fact that aggregations were not analysed here, existing works, where two regions can also be observed in the probability density function of returns [3], points that this distribution will relax firstly to the q -Gaussian [6] and finally to the Gaussian distribution [4]. Although the study of tick-by-tick time series is usually neglected in favour of, e.g., 1 minute lag time series, its analysis could be important in the reach of a better understanding of the nature of these (beautiful financial) systems, because all phenomena observed at major time scales have their origin in this time scale dynamics.

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